Engineering Notes

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Identity Between INS Position and Velocity Error Equations in the True Frame

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Nomenclature

C_s^k = transformation matrix from s to k
f^k = column matrix of specific-force vector coordinatized in k
g^k = column matrix of gravity vector coordinatized in k
g_m^k = column matrix of gravitation vector coordinatized in k
r^k = column matrix of position vector coordinatized in k
v^k = column matrix of Earth-referenced velocity vector coordinatized in k
δ = perturbation in the quantity immediately following δ
ω_{sk}^u = angular rate vector of the k frame with respect to the s frame coordinatized in u
Ω_{sk}^u = skew symmetric matrix of the cross product of ω_{sk}^u

Coordinate frames

c = computer coordinate frame
e = Earth-fixed coordinate frame
i = inertial coordinate frame
k,s,u = general coordinate frames
t = true coordinate frame

Introduction

THE basic physical law used to solve the terrestrial INS navigation equations is

$$\ddot{r}^i = C_s^i[f^s + g_m^s] \tag{1}$$

Utilizing the mathematical rules of differentiation with respect to time in rotating coordinate systems it can be shown that

$$\ddot{r}^i = C_s^i[\ddot{r}^s + 2\Omega_{is}^s\dot{r}^s + \dot{\Omega}_{is}^s r^s + \Omega_{is}^s\Omega_{is}^s r^s] \tag{2}$$

Using Eq. (2) and the following definition

$$g^s \triangleq g_m^s - \Omega_{ie}^s\Omega_{ie}^s r^s \tag{3}$$

in Eq. (1), it can be shown that

$$\ddot{r}^s + 2\Omega_{is}^s\dot{r}^s + [\dot{\Omega}_{is}^s + \Omega_{is}^s\Omega_{is}^s - \Omega_{ie}^s\Omega_{ie}^s]r^s = f^s + g^s \tag{4a}$$

It can also be shown that

$$v^s = \dot{r}^s + \Omega_{es}^s r^s \tag{4b}$$

where the following definition of Earth-referenced velocity is used

$$v^s \triangleq C_e^s\dot{r}^e \tag{5}$$

Equations (4a) and (4b) are the nominal equations whose solution yields the system position and velocity, respectively.[1,2] Equivalently, the following pair[3,4] can be solved as follows:

$$\dot{v}^s + [\Omega_{ie}^s + \Omega_{is}^s]v^s = f^s + g^s \tag{6a}$$

$$\dot{r}^s + \Omega_{es}^s r^s = v^s \tag{6b}$$

where Eq. (6a) is derived from Eq. (1) using the same differentiation rules. The solution of Eq. (6b) yields the position in the reference coordinate system. Of more interest, though, in a terrestrial trajectory, is the Earth-frame position, and for that reason the following equation[5]

$$\dot{r}^e = C_s^e v^s \tag{6c}$$

usually is used instead of Eq. (6b).

We will refer to Eqs. (4a) and (4b) as the *position navigation equations* and to Eqs. (6a–6c) as the *velocity navigation equations*.

The models that describe the translatory error propagation in terrestrial INS stem from two approaches: the computer frame (or psi angle) approach,[1,2] and the true frame (or perturbation) approach.[3,4] Each approach can be applied to either the position or velocity navigation equations. Consequently, there are four possible error models: the computer-frame position error model, the computer-frame velocity error model, the true-frame position error model, and the true-frame velocity error model.

These models are derived through perturbations in different ways which involve omissions of nonlinear terms. A question arises then whether all models are *mathematically* equivalent; that is, can one arrive by proper mathematical operations from one model to another. Benson[5] has shown the equivalence between the computer-frame velocity error model and the true-frame velocity error model; the equivalence of the computer-frame position error model and the computer-frame velocity error model has been shown in Ref. 6. The purpose of this Note is to prove analytically the equivalence between the true-frame position error model and the true-frame velocity error model.

A Brief Review of the Error Models

Although we are concerned here only with the true-frame position error model and the true-frame velocity error model, for the sake of completeness we will present a brief development of all four models.

The Two Computer-frame Error Models

A computer-frame error model is obtained when nominal equations are perturbed in the local-level north-pointing coordinate system, which corresponds to the geographic location

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computed and, hence indicated by the INS (c frame). As a result, the angular rates and the transformation to and from the c frame are known precisely and need not be perturbed in the derivation of the corresponding error equation.

The position error model is obtained from the perturbation of Eqs. (4a) and (4b), in which s (which is an arbitrary coordinate system) is set to be c:

$$\delta\ddot{r}^c + 2\Omega^c_{ic}\delta\dot{r}^c + [\dot{\Omega}^c_{ic} + \Omega^c_{ic}\Omega^c_{ic} - \Omega^c_{ie}\Omega^c_{ie}]\delta r^c = \delta f^c + dg^c \tag{7a}$$

$$\delta v^c = \delta\dot{r}^c + \Omega^c_{ec}\delta r^c \tag{7b}$$

The velocity error model is obtained from Eqs. (6a) and (6c) as follows:

$$\delta\dot{v}^c + [\Omega^c_{ie} + \Omega^c_{ic}]\delta v^c = \delta f^c + \delta g^c \tag{8a}$$

$$\delta\dot{r}^e = C^e_c\delta v^c \tag{8b}$$

The Two True-frame Error Models

A true-frame error model is obtained when nominal equations are perturbed in the local-level north-pointing coordinate system, which corresponds to the point at which the INS is truly located. Here there is no reason to assume a perfect knowledge of any element of the nominal equation, and so all of them have to be perturbed. Perturbing Eqs. (4a) and (6c), in which s is set to be t, yields the following position error model in the true-frame:

$$\begin{aligned}&\delta\ddot{r}^t + 2\Omega^t_{it}\delta\dot{r}^t + 2\delta\Omega^t_{it}\dot{r}^t + \dot{\Omega}^t_{it}\delta r^t + \delta\dot{\Omega}^t_{it}r^t + \Omega^t_{it}\Omega^t_{it}\delta r^t\\ &+\Omega^t_{it}\delta\Omega^t_{it}r^t + \delta\Omega^t_{it}\Omega^t_{it}r^t - \Omega^t_{ie}\Omega^t_{ie}\delta r^t - \Omega^t_{ie}\delta\Omega^t_{ie}r^t\\ &-\delta\Omega^t_{ie}\Omega^t_{ie}r^t = \delta f^t + \delta g^t\end{aligned} \tag{9a}$$

$$\delta v^t = C^t_e\delta\dot{r}^e + \delta C^t_e\dot{r}^e \tag{9b}$$

Perturbing Eqs. (6a) and (6c) and, again, setting s to be t, yields the velocity error model

$$\delta\dot{v}^t + [\Omega^t_{ie} + \Omega^t_{it}]\delta v^t + [\delta\Omega^t_{ie} + \delta\Omega^t_{it}]v^t = \delta f^t + \delta g^t \tag{10a}$$

$$\delta\dot{r}^e = C^e_t\delta v^t + \delta C^e_t v^t \tag{10b}$$

Equivalence Proof

As stated in the introduction, our purpose is to prove the equivalence between the true-frame position error model in Eq. (9a) and the true-frame velocity error model given in Eq. (10a). To meet this end we start with Eq. (9a) and, using known mathematical and kinematic relations, derive Eq. (10a). However, before doing so we develop an auxiliary result, which will be given in Eq. (17).

When, in Eq. (6b), s is chosen to be t, the following is obtained:

$$\dot{r}^t = v^t - \Omega^t_{et}r^t \tag{11}$$

which, when perturbed, yields

$$\delta\dot{r}^t = \delta v^t - \Omega^t_{et}\delta r^t - \delta\Omega^t_{et}r^t \tag{12}$$

Differentiating Eq. (12) with respect to time in the true-frame results in

$$\delta\ddot{r}^t = \delta\dot{v}^t - \dot{\Omega}^t_{et}\delta r^t - \Omega^t_{et}\delta\dot{r}^t - \delta\dot{\Omega}^t_{et}r^t - \delta\Omega^t_{et}\dot{r}^t \tag{13}$$

Since

$$\omega^s_{et} = \omega^s_{it} - \omega^s_{ie} \tag{14}$$

then, obviously

$$\Omega^t_{et} = \Omega^t_{it} - \Omega^t_{ie} \tag{15}$$

Consequently

$$-\dot{\Omega}^t_{et} = -\dot{\Omega}^t_{it} + \dot{\Omega}^t_{ie} \tag{16a}$$

and

$$-\delta\dot{\Omega}^t_{et} = -\delta\dot{\Omega}^t_{it} + \delta\dot{\Omega}^t_{ie} \tag{16b}$$

Substituting Eqs. (16) into Eq. (13) yields

$$\delta\ddot{r}^t = \delta\dot{v}^t + [\dot{\Omega}^t_{ie} - \dot{\Omega}^t_{it}]\delta r^t - \Omega^t_{et}\delta\dot{r}^t + [\delta\dot{\Omega}^t_{ie} - \delta\dot{\Omega}^t_{it}]r^t - \delta\Omega^t_{et}\dot{r}^t \tag{17}$$

which is the auxiliary result needed for the proof. Now, substituting this result into Eq. (9a) and using Eq. (15) yields

$$\begin{aligned}&\delta\dot{v}^t + [2\Omega^t_{ie} + \Omega^t_{et}]\delta\dot{r}^t + [\dot{\Omega}^t_{ie} + \Omega^t_{it}\Omega^t_{it} - \Omega^t_{ie}\Omega^t_{ie}]\delta r^t\\ &+[2\delta\Omega^t_{ie} + \delta\Omega^t_{et}]\dot{r}^t + [\delta\dot{\Omega}^t_{ie} + \delta(\Omega^t_{it}\Omega^t_{it} - \Omega^t_{ie}\Omega^t_{ie})]r^t\\ &= \delta f^t + \delta g^t\end{aligned} \tag{18}$$

where in arriving at Eq. (18) we used the obvious relation

$$\delta\Omega^t_{it}\Omega^t_{it} + \Omega^t_{it}\delta\Omega_{it} - \delta\Omega^t_{ie}\Omega^t_{ie} - \Omega^t_{ie}\delta\Omega^t_{ie} = \delta(\Omega^t_{it}\Omega^t_{it} - \Omega^t_{ie}\Omega^t_{ie}) \tag{19}$$

It can be shown [e.g., Eqs. (8–99) in Ref. 4] that

$$C^t_e\dot{\Omega}^e_{ie}C^e_t = \dot{\Omega}^t_{ie} + \Omega^t_{it}\Omega^t_{ie} - \Omega^t_{ie}\Omega^t_{it} \tag{20}$$

Noting that Ω^e_{ie} consists of constant Earth-rate components, and using Eq. (15), the last equation can be written as

$$\dot{\Omega}^t_{ie} = -\Omega^t_{et}\Omega^t_{ie} + \Omega^t_{ie}\Omega^t_{et} \tag{21}$$

hence

$$\delta\dot{\Omega}^t_{ie} = -\delta(\Omega^t_{et}\Omega^t_{ie}) + \delta(\Omega^t_{ie}\Omega^t_{et}) \tag{22}$$

Then, substituting Eqs. (21), (22), (11), and (12) into Eq. (18) and rearranging terms results in

$$\begin{aligned}&\delta\dot{v}^t + [2\Omega^t_{ie} + \Omega^t_{et}]\delta v^t + [\Omega^t_{it}\Omega^t_{it} - \Omega^t_{ie}\Omega^t_{ie} - \Omega^t_{et}\Omega^t_{ie}\\ &-\Omega^t_{ie}\Omega^t_{et} - \Omega^t_{et}\Omega^t_{et}]\delta r^t + [2\delta\Omega^t_{ie} + \delta\Omega^t_{et}]v^t\\ &+[\delta(\Omega^t_{it}\Omega^t_{it}) - \delta(\Omega^t_{ie}\Omega^t_{ie}) - \delta(\Omega^t_{et}\Omega^t_{ie}) - \delta(\Omega^t_{ie}\Omega^t_{et})\\ &-\delta(\Omega^t_{et}\Omega^t_{et})]r^t = \delta f^t + \delta g^t\end{aligned} \tag{23}$$

Using Eq. (15) it is evident that

$$\Omega^t_{it}\Omega^t_{it} = \Omega^t_{ie}\Omega^t_{ie} + \Omega^t_{et}\Omega^t_{ie} + \Omega^t_{ie}\Omega^t_{et} + \Omega^t_{et}\Omega^t_{et} \tag{24a}$$

and hence

$$\delta(\Omega^t_{it}\Omega^t_{it}) = \delta(\Omega^t_{ie}\Omega^t_{ie}) + \delta(\Omega^t_{et}\Omega^t_{ie}) + \delta(\Omega^t_{ie}\Omega^t_{et}) + \delta(\Omega^t_{et}\Omega^t_{et}) \tag{24b}$$

Substituting the last two results into Eq. (23) zeroes the bracketed terms, premultiplying δr^t and r^t, respectively. This results in

$$\delta\dot{v}^t + [2\Omega^t_{ie} + \Omega^t_{et}]\delta v^t + [2\delta\Omega^t_{ie} + \delta\Omega^t_{et}]v^t = \delta f^t + \delta g^t \tag{25}$$

which is Eq. (10a). This completes the proof.

References

[1]Leondes, C. T. (ed.), *Guidance and Control of Aerospace Vehicles*, McGraw-Hill, New York, 1963, Chap. 2.

[2]Pitman, G. R., Jr. (ed.), *Inertial Guidance*, Wiley, New York, 1962.

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[4]Britting, K. R., *Inertial Navigation Analysis*, Wiley Interscience, New York, 1971.

[5]Benson, D. O., Jr., "A Comparison of Two Approaches to Pure-Inertial and Doppler-Inertial Error Analysis," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. AES-11, July 1975, pp. 447-455.

[6]Bar-Itzhack, I. Y., "Identity Between INS Position and Velocity Error Models," *Journal of Guidance and Control*, Vol. 4, Sept.–Oct. 1981, pp. 568–570.

Singular Perturbation and Time Scale Approaches in Discrete Control Systems

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Introduction

THE theory of singular perturbations and time scales has been a powerful analytical tool in the analysis and synthesis of continuous and discrete control systems.[1,2] In this Note, we first consider a singularly perturbed discrete control system. Using a singular perturbation approach, outer and correction subsystems are obtained. Next, by the application of a time scale approach via block diagonalization transformations, the original system is decoupled into slow and fast subsystems. It will be shown that, to a zeroth-order approximation, the singular perturation and time scale approaches yield equivalent results. Roughly speaking, the zeroth-order approximation is sometimes called the first approximation. This result is similar to a corresponding result in continuous control systems.[3]

Singular Perturbation Approach

Consider a general form for linear, shift-invariant, singularly perturbed discrete systems as[2]

$$x(k+1)=A_{11}x(k)+h^{1-j}A_{12}z(k)+B_1u(k) \tag{1a}$$

$$h^{2i}z(k+1)=h^jA_{21}x(k)+hA_{22}z(k)+h^jB_2u(k) \tag{1b}$$

$$0\leq i\leq 1;\ 0\leq j\leq 1$$

where $x(k)$ and $z(k)$ are "slow" and "fast" state vectors of n and m dimensions, respectively, $u(k)$ an r-dimensional control vector, h a singular perturbation parameter, and A and B matrices of appropriate dimensionality. We formulate initial value problems with $x(k=0)=x(0)$ and $z(k=0)=z(0)$ and note that similar results can be obtained for boundary value problems as well.

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The three limiting cases of Eq. (1) result in the following models:

1) The C-model ($i=0$, $j=0$),

$$x(k+1)=A_{11}x(k)+hA_{12}z(k)+B_1u(k) \tag{2a}$$

$$z(k+1)=A_{21}x(k)+hA_{22}z(k)+B_2u(k) \tag{2b}$$

where the small parameter h appears in the *column* of the system matrix.

2) The R-model ($i=0$, $j=1$).

$$x(k+1)=A_{11}x(k)+A_{12}z(k)+B_1u(k) \tag{3a}$$

$$z(k+1)=hA_{21}x(k)+hA_{22}z(k)+hB_2u(k) \tag{3b}$$

where the small parameter h appears in the *row* of the system matrix.

3) The D-model ($i=1$, $j=1$),

$$x(k+1)=A_{11}x(k)+A_{12}z(k)+B_1u(k) \tag{4a}$$

$$hz(k+1)=A_{21}x(k)+A_{22}z(k)+B_2u(k) \tag{4b}$$

where the small parameter h is positioned in an identical fashion to that of the continuous systems described by *differential* equations. In this Note, we consider only the C-model of Eq. (2), but the result can be extended to the other two models of Eqs. (3) and (4) as well. The outer (degenerate) subsystem, obtained by zeroth-order approximation (i.e., by making $h=0$) of Eq. (2), is

$$x^{(o)}(k+1)=A_{11}x^{(o)}(k)+B_1u^{(o)}(k) \tag{5a}$$

$$z^{(o)}(k+1)=A_{21}x^{(o)}(k)+B_2u^{(o)}(k) \tag{5b}$$

$$x^{(o)}(k=0)=x(0);\ z^{(o)}(k=0)\neq z(0) \tag{5c}$$

Here, we note that in the process of degeneration, $x(k)$ has retained its initial condition $x(0)$, whereas $z(k)$ has lost its initial condition $z(0)$. In order to recover this lost initial condition, a correction subsystem is used.[2] The transformations between the original and correction variables are

$$x_c(k)=x(k)/h^{k+1};\ z_c(k)=z(k)/h^k \tag{6a}$$

$$u_c(k)=u(k)/h^{k+1} \tag{6b}$$

Using Eq. (6) in Eq. (2), the transformed system becomes

$$hx_c(k+1)=A_{11}x_c(k)+A_{12}z_c(k)+B_1u_c(k) \tag{7a}$$

$$z_c(k+1)=A_{21}x_c(k)+A_{22}z_c(k)+B_2u_c(k) \tag{7b}$$

The zeroth-order approximation ($h=0$) of Eq. (7) becomes

$$0=A_{11}x_c{}^{(o)}(k)+A_{12}z_c{}^{(o)}(k)+B_1u_c{}^{(o)}(k) \tag{8a}$$

$$z_c{}^{(o)}(k+1)=A_{21}x_c{}^{(o)}(k)+A_{22}z_c{}^{(o)}(k)+B_2u^{(o)}(k) \tag{8b}$$

Rewriting Eq. (8), we get

$$x_c{}^{(o)}(k)=-A_{11}{}^{-1}[A_{12}z_c{}^{(o)}(k)+B_1u_c{}^{(o)}(k)] \tag{9a}$$

$$z_c{}^{(o)}(k+1)=A_{c0}z_c{}^{(o)}(k)+B_{c0}u_c{}^{(o)}(k) \tag{9b}$$

where

$$A_{c0}=A_{22}-A_{21}A_{11}{}^{-1}A_{12}$$

$$B_{c0}=B_2-A_{21}A_{11}{}^{-1}B_1$$